Parametric Relations

(1)Find the length of the curve given by:

(i) $x=t^2,\ y=t^3,\ 0\le t\le 2$ (ii) $x=\frac{3}{2}t^2+1,\quad y=2t^2+3,\ 2\le t\le 3$

(iii) $x=1+t,\quad y=1+\cosh t,\ 0\le t\le 2$ (iv) $x=\cos\ t,\ y=\ t\sin\ t,\quad t\ \ in\,[0,\ \pi]$

(2)Find the area of the region bounded by the curve, x-axis:

(i) $x = 3(t - \sin t)$, $y = 3(1 - \cos t)$, t in $[0, 2\pi]$.

(ii) $x=2+t^3,\quad y=1+t^2,\ 0\le t\le 1.$

(iii) $x = t + \sin t$, $y = 1 + \cos t$, t in $[0, \pi]$.

(3)Find the volume of the solid generated by revolving, the region between the curve:

(a) $x = \cos t$, $y = \sin t$, t in $[0, \pi/2]$, x- axis, about (i) x-axis (ii) y-axis

(b) $x = 1 + \sin t$, $y = 1+ \cos t$, t in $[0, 2\pi]$, x- axis, about (i) x-axis (ii) y-axis

(4)Find surface area of the surface generated by rotating, about x-axis, the curve:

(i) $x = \cos t$, $y = 2 + \sin t$, t in $[0, 2\pi]$ (ii) $x=t^2+2,\ y=t,\ 2\le t\le 3$

Vector Analysis

(1) If $\phi=2xz^4-x^2y$ find $\vec{\nabla}\phi,\left|\vec{\nabla}\phi\right|$ at the point $(2,-2,1)$.[Ans. $10\vec{i}-4\vec{j}-\overrightarrow{16k},\ 2\sqrt{93}$]

(2) Find $\vec{\nabla}\left|\vec{r}\right|^3$ where $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$. [Ans.: $3\left|\vec{r}\right|\vec{r}$].

(3) Find $\phi(x,y,z)$ which satisfy $\vec{\nabla}\phi=2xyz^3\,\vec{i}+x^2z^3\,\vec{j}+3x^2yz^2\,\vec{k}$,

$\phi(1,-2,2)=4$ [Ans.: $\phi=x^2yz^3+20$].

(4) Find the unit vector normal to $(x-1)^2+y^2+(z+2)^2=9$ at the point $(3,1,-4)$.

[Ans.: $\dfrac{2\vec{i}+\vec{j}-2\vec{k}}{3}$].

(5) Evaluate $\int_0^2 A(t)dt$ if $A(t)=(3t^2-t)\vec{i}+(2-6t)\vec{j}-4t\,\vec{k}$.

(6) Evaluate $\int_0^{2\pi}(3\sin t\,\vec{i}+2\cos t\,\vec{j})dt$. [Ans.: $3\vec{i}+2\vec{j}$].

(7) Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}(t)=(5xy-6x^2)\vec{i}+(2y-4x)\vec{j}$ along the curve

$C: y = x^3$ from the point $(1,1)$ to the point $(2,8)$. [Ans.: 35].

(8) Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (2x+y)\vec{i} + (3y-x)\vec{j}$ along the curve which consists of the straight lines from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$. [11].

(9) Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (x-3y)\vec{i} + (y-2x)\vec{j}$ and C is the closed curve $x = 3\sin\theta,\ \ y = 2\cos\theta$ in xy – plane [Ans.: 6π].

(10) Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (2x+y^2)\vec{i} + (3y-4x)\vec{j}$ and C is the closed curve connected the points $(0,0),(2,0),(2,1)$ taken in positive direction: $[\frac{-14}{3}]$.

(11) Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$ and C is the closed curve in xy – plane consisting of $y = x^2,\ x = y^2$. [Ans.: $\frac{2}{3}$]

(12) If $\vec{F} = (4yx - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x3z\vec{k}$

(i) prove that $\int_C \vec{F}.d\vec{r}$ independent to any path through tow any point .

(ii) Find φ such that $\vec{F} = \vec{\nabla}\varphi$. [Ans.: $\varphi = 2x^2y - x^3z^2 + c$].

(13) (i) Show that $\vec{F} = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$ is a force of conservative field.

(ii) Find the scalar field [Ans.: $\varphi = y^2\sin x + z^3x - 4y + 2z + c$].

(iii) Find the work done to transfer a body from $(0,1,-1)$ to $(\frac{\pi}{2},-1,2)$.

(14) Verify **Green's theorem** in the plane for the integral

$\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)dy$ Where C the region enclosed by:

$x + y = 1,\ \ x = 0,\ \ y = 0$ [Ans. $\frac{5}{3}$]

(15) Evaluate $\oint_C (3x+4y)dx+(2x-3y)dy$ where C is the circle $x^2+y^2=4$ $[-8\pi]$

(16) Evaluate $\oint (x^2+y^2)dx+(3xy^2)dy$ where C is the circle $x^2+y^2=4$ $[12\pi]$

(17) Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy-y^2)dx+(3x^2-2xy)dy$ on the path

$x=t-\sin t,\quad y=1-\cos t.$ [Ans. $6\pi^2-4\pi$]

(18) Evaluate $\oint_C (3x^2+2y)dx-(x+3\cos y)dy$ around the circumference of the

triangle which its vertices at $(1,1), (2,0), (0,0)$.[Ans. -6]

(19) Calculate the area bounded by the curve

$x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta),\ a>0,\quad 0\le\theta\le 2\pi$ and x-axes. [Ans. $3\pi a^2$]

(20) Find the area enclosed by the following curves: $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},\ a>0$ [$\frac{3\pi a^2}{8}$]

Hint the parametric equation is $x=a\cos^3 t,\quad y=a\sin^3 t$

(21) Verify **Green's theorem** in the plane for the integral

$\oint_C (2x-y^3)\,dx-xy\,dy$ where C the region bounded by

$x^2+y^2=1,\quad x^2+y^2=9.$ [Ans. 60π]

Fourier Analysis

(I) Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π

(1) $f(x)=x \quad -\pi<x<\pi$ $\quad$ *Ans* : $\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}\sin nx}{n}$

(2) $f(x)=\begin{cases}-x & -\pi<x<0 \\ x & 0<x<\pi\end{cases}$ $\quad$ *Ans* : $\pi+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{-4\cos(2n-1)x}{(2n-1)^2}$

(3) $f(x)=x^2 \quad -\pi<x<\pi$

(4) $f(x)=\begin{cases}0 & -\pi<x<0 \\ 1 & 0<x<\pi\end{cases}$ $\quad$ *Ans* : $\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}$

(II) Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π.

(1) $f(x)=x^3, \quad -\pi<x<\pi$

(2) $f(x)=|x|, \quad -\pi<x<\pi$

(3) $f(x)=|\sin x|, \quad -\pi<x<\pi$

(4) $f(x)=\begin{cases}-x^2, & -\pi<x<0 \\ x^2, & 0<x<\pi\end{cases}$

(5) $f(x)=\begin{cases}-\cos x, & -\pi<x<0 \\ \cos x, & 0<x<\pi\end{cases}$

(6) show that $f(x)=x^2, -\pi<x<\pi, f(x)=f(x+2\pi)$

has the Fourier series $f(x)=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2}$

Hence show that $(i)\ \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ $\qquad (ii)\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

(III) Represent the following function $f(x)$ by a Fourier cosine series

(1) $f(x) = x, \quad 0 < x < \pi$

(2) $f(x) = x^2, \quad 0 < x < \pi$

(3) $f(x) = \sin x, \quad 0 < x < \pi$ $\quad$ *Ans.* $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(4n^2 - 1)}$

(4) $f(t) = \begin{cases} 1, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$ $\quad$ *Ans.* $\frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} \cos(2n-1)\pi x$

(IV) Represent the following function $f(t)$ by a Fourier sine series

(1) $f(x) = 1, \quad 0 < x < \pi$

(2) $f(x) = x, \quad 0 < x < \pi$

(3) $f(x) = x^2, \quad 0 < x < \pi$

(4) $f(x) = \begin{cases} 0, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$

Complex Analysis

(1) Express the following function in the form $u(x,y) + iv(x,y)$

$(a)\ w = z^3 \qquad (b)\ w = 2z^{-2} + 1 \qquad (c)\ w = \frac{1}{z}$

$(d)\ w = \frac{z}{1+z} \qquad (e)\ w = z + \frac{1}{z} \qquad (f)\ w = z\bar{z}$

$(g)\ \exp \bar{z}, \qquad (h) \cos iz, \qquad (i) \sin i\bar{z}$

(2) Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiable at any point z.

(3) Show that the following functions satisfy Cauchy Riemann equations

$(a)\ f(z) = iz^2 + 2z \qquad (b)\ f(z) = \sin z \qquad (c)\ f(z) = ze^{-z}$

(4) Prove that **(i)** $\overline{\sin z} = \sin \bar{z}$ $\qquad$ **(ii)** $\overline{\cos z} = \cos \bar{z}$

(5) By using Cauchy integral Formula evaluate the following integrals:

(a) $\oint_C \frac{z^2+5}{z-3}dz$ where C is the circle $|z|=4$.

(b) $\oint_C \frac{z\,dz}{z^2-1}$ where C is the circle $|z|=2$. (Ans. $2\pi i$)

(c) $\oint_C \frac{e^z}{z^2+4}dz$ where C is the circle $|z-i|=2$. (Ans. $\frac{\pi i}{4}$)

(d) $\oint_C \frac{zdz}{(9-z^2)(z+i)}$ where C is the circle $|z|=2$.

(e) $\oint_C \frac{z^2+5}{z-3}dz$ where C is the circle $|z-1|=4$.

(6) Prove that

(a) $\oint_C \frac{\sin^2 z}{(z-\frac{\pi}{6})^3}dz = \pi i$ where C is the circle $|z|=1$.

(b) $\oint_C \frac{\sin z}{(z-\frac{\pi}{2})^2}dz = 0$ where C is the circle $|z|=2$.

(c) $\oint_C \frac{e^{2z}}{(z+1)^4}dz = \left(\frac{i8\pi e^{-2}}{3}\right)$ where C is the circle $|z|=2$.